METRIC DIOPHANTINE APPROXIMATION WITH RESPECT TO PLANAR DISTANCE FUNCTIONS

SIMON KRISTENSEN (EDINBURGH)

ABSTRACT. We outline a proof of an analogue of Khintchine's Theorem in $\mathbb{R}^2$, where the ordinary height is replaced by a distance function satisfying an irrationality condition as well as certain decay and symmetry conditions.

1. INTRODUCTION

A star body in the plane $S \subseteq \mathbb{R}^2$ is defined as an open set which has the origin as an interior point, such that for any $\mathbf{x} \in \mathbb{R}^2$, there exists a $t_0 \in (0, \infty]$ such that for $t < t_0$, $t\mathbf{x} \in S$ and for $t > t_0$, $t\mathbf{x} \notin S$ (see [3]). To such sets, one can associate a distance function $F : \mathbb{R}^2 \to \mathbb{R}$ such that $F(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^2$, and such that for any $\mathbf{x} \in \mathbb{R}^2$ and any $t \geq 0$, $F(t\mathbf{x}) = tF(\mathbf{x})$. The open star body S may be expressed as the set of points satisfying $F(\mathbf{x}) < 1$. Conversely, to each continuous distance function, we may associate an open star body through this relation. Throughout this paper, vectors will be denoted in bold typeface and their coordinates in normal typeface with a subscript, *e.g.*, $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$.

Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a distance function and let $\psi : \mathbb{N} \to \mathbb{R}_+$ be a function such that $q\psi(q)$ is non-increasing. The measure of the set

(1) $\quad W(F; \psi) = \big\{ \mathbf{x} \in [0,1)^2 : \text{ for infinitely many } q \in \mathbb{N},$

$$\text{there is a } \mathbf{p} \in \mathbb{Z}^2, \text{ such that } F(\mathbf{x} - \mathbf{p}/q) < \psi(q) \big\},$$

has been studied for a number of distance functions F, the most important of which are the height $\max\{|x_1|, |x_2|\}$ and the multiplicative distance function $\sqrt{|x_1| |x_2|}$. For both of these, the measure is null or full accordingly as a certain series depending on F and ψ converges or diverges. For the height, the series is $\sum_{q=1}^{\infty} q^2 \psi(q)^2$. This was shown by Khintchine [9]. For the multiplicative distance function, the series is $\sum_{q=1}^{\infty} q^2 \psi(q)^2 \log(q^{-1}\psi(q)^{-1})$. This result is a corollary of a result by Gallagher [7].

Note that in the original statements of theorems of Khintchine and Gallagher, the sets are defined slightly differently. Originally, $\mathbf{p}$ is the integral vector closest to $q\mathbf{x}$. However, using the form of the distance functions, it may easily be shown that the sets originally studied by Khintchine and Gallagher are equal to $W(F; \psi)$ for the appropriate distance function.

The Hausdorff dimension of the set $W(F; \psi)$ was studied by Dodson and Hasan [4, 5] under a technical covering condition. When the dimension is less than 2, the set is obviously null. Unfortunately, this covering condition is very difficult to check for most non-trivial star bodies. Also, their results do not provide conditions under which the measure of the set is positive.

We will outline a proof of the fact that for a class of distance functions, the measure of $W(F; \psi)$ is full regardless of the error function ψ. We will make a number of technical assumptions on the distance function, some of which are unnecessary in general, but which will make the present presentation simpler. The full result will be published elsewhere in a joint paper with M. M. Dodson [6] where additional distance functions will also be treated.

Let $F : \mathbb{R}^2 \to \mathbb{R}_+$ be a distance function satisfying the following conditions:

(1) The set $F^{-1}(0)$ consists of a family of half-lines such that at least one has an irrational slope.

(2) One irrationally sloped half-line L from $F^{-1}(0)$ satisfies that for any $M > 0$,

$$\mu\left(\left\{\mathbf{x} \in \mathbb{R}^2 : F(\mathbf{x}) < 1\right\} \cap \left\{\mathbf{x} \in \mathbb{R}^2 : \mathrm{dist}(\mathbf{x}, L) < M\right\}\right) = \infty.$$

(3) For any point $\mathbf{x}$ on the half-line L with $|\mathbf{x}|$ sufficiently large, the line orthogonal to L through $\mathbf{x}$ intersects the level curve $F(\mathbf{x}) = 1$ in such a way that the two intersection points closest to $\mathbf{x}$ are at equal distance from $\mathbf{x}$.

(4) The distance from L to the closest intersection point of the line orthogonal to L through the point $\mathbf{x}$ is decreasing as a function of $|\mathbf{x}|$ for $|\mathbf{x}|$ large enough.

Broadly speaking, the conditions mean that the star body has (1) an unbounded component about an irrationally sloped half line, (2) such that this component has infinite measure, (3) such that the unbounded component is symmetric about the irrationally sloped line and (4) such that the width of the star body decreases as we move along the unbounded component. Condition (3) is strictly speaking not needed, but it will make the present presentation easier to follow. The full result without the additional requirements will be proved in [6].

We now state the main result.

Theorem 1. *Let $F : \mathbb{R}^2 \to \mathbb{R}$ be a distance function satisfying conditions (1)–(4). Then, for any $\psi : \mathbb{N} \to \mathbb{R}_+$, for any $q \in \mathbb{N}$ and for almost all $\mathbf{x} \in [0, 1)^2$ there are infinitely many $\mathbf{p} \in \mathbb{Z}^2$ such that*

$$(2) \qquad\qquad\qquad\qquad F(\mathbf{x} - \mathbf{p}/q) < \psi(q).$$

In particular,

$$|W(F; \psi)| = 1.$$

This result extends Khintchine's Theorem in a rather surprising way to a larger class of distance functions not covered by any of the previous results. Gallagher's result [7] allows us to deal with distance functions where $F^{-1}(0)$ may contain any of the coordinate half-axes but no other half-lines. Furthermore, as the measure is always full, the results of Dodson and Hasan are not applicable to these situations.

2. Outline of proof

We now outline the proof of Theorem 1. Let $q \in \mathbb{N}$ be fixed and let α denote the slope of the line L satisfying assumptions (1)–(4). Suppose that $\alpha \geq 1/2$ and that L is in the first quadrant. This causes no loss of generality, as we may change the role of the axes if this is not the case to obtain a similar situation. The family of lines $\mathcal{F} = \{L + \mathbf{p}/q : \mathbf{p} \in \mathbb{Z}^2\}$ induces a family of dense geodesics on the standard torus $\mathbb{T}^2$ having $[0,1)^2$ as its fundamental domain. This follows from Kronecker's Theorem (see *e.g.* [8, Proposition 1.5.1]).

We fix one such geodesic and consider a fixed horizontal line across $[0,1)^2$. With the endpoints identified, this may be identified with a unit circle $\mathbb{S}$. Furthermore, the geodesic on $\mathbb{T}^2$ intersects this circle at multiples of α^{-1} plus some constant corresponding to the first intersection y_0. That is, we have points $\{y_0 + n\alpha^{-1}\}$, where $\{x\}$ denotes the fractional part of x. These points are uniformly distributed, but this is unfortunately not sufficient to ensure that the measure of the intersection between $W(F; \psi)$ and the circle is full.

We now study the intersection of the unbounded branch of the star bodies centred at the points $\mathbf{p}/q$ and the circle. About the geodesic, there is a strip which becomes increasingly narrow as we move along. The width of this strip is decreasing by assumption (4). Considering the intersection with $\mathbb{S}$, we find that to each point $\{y_0 + n\alpha^{-1}\}$ corresponds an interval of some length. We consider subintervals of these centred at $\{y_0 + n\alpha^{-1}\}$ of width ρ_n comparable to the width of the strip at a point along L at some multiple of n. That ρ_n is comparable to the width follows from assumption (3). Finally, assumption (2) along with the homogeneity of the distance function implies that $\sum_{n=1}^{\infty} \rho_n = \infty$ regardless of the error function ψ.

We have now reduced the problem of finding the measure of the set of points $\mathbf{x}$ on the horizontal line chosen for which $F(\mathbf{x} - \mathbf{p}/q) < \psi(q)$ for infinitely many $\mathbf{p} \in \mathbb{Z}^2$ to the problem of finding the measure of the set of points $x \in \mathbb{S}$ satisfying the inequalities

$$(3) \qquad \left\| n\alpha^{-1} + y_0 - x \right\| < \rho_n$$

for infinitely many $n \in \mathbb{N}$. Indeed, such points will satisfy the original inequalities by the above arguments.

When the series $\sum \rho_n$ converges, the Hausdorff dimension of the set of points $y_0 - x$ satisfying (3) infinitely often was found by Bugeaud [2] using regular systems. His arguments may be adapted to deal with the situation when the series is divergent. For this purpose, the notion of a locally ubiquitous system is required.

Let $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ be a function decreasing to zero and let (N_r) be a strictly increasing sequence of natural numbers. The set of points $\mathcal{R} = \{z_n : n \in \mathbb{N}\}$ in the metric space $\mathcal{S}$ is said to be *locally ubiquitous relative to λ and (N_r)* if there is a constant $\kappa > 0$ such that for any interval $I = (c - \rho, c + \rho)$ with ρ sufficiently small and any $r \geq r_0(I)$,

$$\mu\left(I \cap \bigcup_{N_r \leq n < N_{r+1}} (z_n - \lambda(N_r), z_n + \lambda(N_r))\right) \geq 2\kappa\rho.$$

This is a quantitative version of the local density of the points in $\mathcal{R}$.

Using the so-called Three Distances Theorem due to Sós [10], we may easily prove that the existence of a strictly increasing sequence (N_r) such that the set of points $\{n\alpha^{-1}+y_0\}$ is locally ubiquitous with respect to λ and (N_r), where $\lambda(N) = 3/(1+N_r)$ for $N \in [N_r, N_{r+1})$. Using a recent result due to Beresnevich, Dickinson and Velani [1], this along with the divergence of $\sum \rho_n$ is sufficient to ensure that the set of $x \in \mathbb{S}$ for which (3) is satisfied for infinitely many $n \in \mathbb{N}$ has full measure.

As we may prove full measure for every horizontal line across $[0, 1)$, the theorem follows on integrating over the relevant characteristic functions in the vertical direction.

3. Concluding remarks

We have only given an outline of the proof of the main theorem. In fact, a more general theorem may be proven, where assumption (3) is no longer needed. In [6] we will prove the more general result in full detail. The ubiquity argument becomes more involved when (3) is not assumed.

In this paper, we will also prove a Khintchine type theorem for distance functions where $F^{-1}(0)$ consists of finitely many half-lines with rational slopes. In this case, the result depends on the error function and takes a form resembling the classical theorem. This shows that the phenomenon studied by the various Khintchine type theorems is very sensitive to minor changes in the distance function. For example, a small rotation of the star body corresponding to the multiplicative distance function may cause a slope to go from rational to irrational, and thus completely changing the nature of the 0–1 law of the associated Khintchine type theorem. This is another example of the wide contrast between the rational and the irrational.

4. Acknowledgements

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SCHOOL OF MATHEMATICS, UNIVERSITY OF EDINBURGH, KING'S BUILDINGS, JCMB, MAYFIELD ROAD, EDINBURGH, EH9 3JZ, SCOTLAND

E-mail address: Simon.Kristensen@ed.ac.uk